Combinatorics of Minkowski decomposition of associahedra

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Abstract

Realisations of associahedra can be obtained from the classical permutahedron by removing some of its facets and the set of these facets is determined by the diagonals of certain labeled convex planar n-gons as shown by Hohlweg and Lange (2007). Ardila, Benedetti, and Doker (2010) expressed polytopes of this type as Minkowski sums and differences of dilated faces of a standard simplex and computed the corresponding coefficients y_I by Möbius inversion. Given an associahedron of Hohlweg and Lange, we give a new combinatorial interpretation of the values y_I: they are the product of two signed lengths of paths of the labeled n-gon. We also discuss an explicit realisation of a cyclohedron to show that that the formula of Ardila, Benedetti, and Doker does not hold for generalised permutahedra not in the deformation cone of the classical permutahderon.

1 Introduction

Consider the convex $(n-1)$-dimensional polytope

$$P_n(\{z_I\}) := \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{smallmatrix} \sum_{i\in[n]} x_i = z_{[n]} \text{ and} \\ \sum_{i\in I} x_i \geq z_I \text{ for } \varnothing\subset I\subset[n] \end{smallmatrix} \right\},$$

where $[n]$ denotes the set $\{1, 2, \cdots, n\}$. The classical permutahedron, as described for example by G. M. Ziegler, [21], corresponds to $z_I = \frac{|I|(|I|+1)}{2}$ for $\varnothing \subset I \subseteq [n]$. *Generalised permutahedra* were first studied by A. Postnikov, [14]. They are polytopes $P_n(\{z_I\})$ and are contained in the deformation cone of the classical permutahedron, [15]. We focus our study on special realisations of associahedra denoted by As^c_{n-1}, which form a subclass of generalised permutahedra. Two examples of 3-dimensional polytopes As^c_3 are shown in Figure 1. In Section 5, we give an example to explain the notion of a deformation cone and to show that the approach to compute the coefficients of the Minkowski decomposition fails for polytopes $P_n(\{z_I\})$ not contained in the deformation cone of the classical permutahedron.

The Minkowski sum of two polytopes P and Q is defined as $\{p + q \mid p \in P,\ q \in Q\}$. On the other hand, we define the Minkowski difference $P - Q$ of polytopes P and Q if and only if there is a polytope R such that $P = Q + R$, for more details on Minkowski differences we refer to [18] . We are interested in decompositions of As^c_{n-1} into Minkowski sums and differences of dilated faces of the $(n-1)$-dimensional standard simplex

$$\Delta_n = \text{conv}\{e_1, e_2, \cdots, e_n\},$$

where e_i is a standard basis vector of $\mathbb{R}^n$. The faces Δ_I of Δ_n are given by $\text{conv}\{e_i\}_{i\in I}$ for $I \subseteq [n]$. If a polytope P is the Minkowski sum and difference of dilated faces of Δ_n, we say that P has a Minkowski decomposition into faces of the standard simplex. The following is a general result on Minkowski decompositions of a generalised permutahedron $P(\{z_I\})$ where we assume that the values z_I for redundant inequalities of $P(\{z_I\})$ are tight.

Proposition 1 ([1, Proposition 2.3])
Every generalised permutahedron $P_n(\{z_I\})$ can be written uniquely as a Minkowski sum and difference of faces of Δ_n:

$$P_n(\{z_I\}) = \sum_{I\subseteq[n]} y_I \Delta_I$$

where $y_I = \sum_{J\subseteq I} (-1)^{|I\setminus J|} z_J$ for each $I \subseteq [n]$.

To put it differently, the functions $I \longmapsto z_I$ and $I \longmapsto y_I$ of the boolean lattice are Möbius inverses. A weaker version of Proposition 1 that requires $y_I \geq 0$ for all $I \subseteq [n]$ was established by A. Postnikov, [14]. Obviously, the formula of Proposition 1 is computationally expensive in general. The formula describes a beautiful relation between the z_I- and y_I-coordinates of generalised permutahedra, but there is more hidden. The author showed that the formula for y_I of Proposition 1

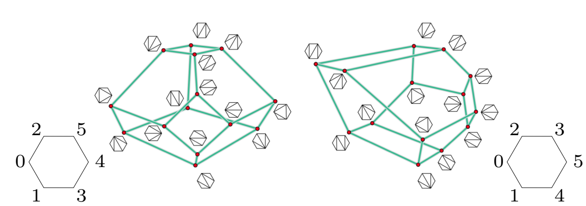

Figure 1: Two different realisations As^c_3 according to [8] after application of an orthogonal transformation. The realisations correspond to different labelings of a hexagon and have distinct Minkowski decompositions into dilated faces of the standard simplex.

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simplifies to four terms for all I if $P(\{z_I\}) = \mathsf{As}^c_{n-1}$, see Theorem 2 and [11]. But even better: we do not even have to compute the four values z_J that remain after simplification, the multiplication of two (signed) numbers of edges connecting points on the boundary of a polygon suffices. The precise statement is given in Theorem 4.

We end this introduction with some general remarks. S. Fomin and A. Zelevinsky introduced generalised associahedra in the context of cluster algebras of finite type, [5], and it is known that associahedra and generalised associahedra associated to cluster algebras of type A are combinatorially equivalent. The construction of [8] was generalised by C. Hohlweg, C. Lange, and H. Thomas to generalised associahedra, [9]. The construction depends on choosing a Coxeter element c and the normal vectors of the facets are determined by combinatorial properties of c. Since the normal fans of these realisations turn out to be Cambrian fans as described by N. Reading and D. Speyer, [16], the obtained realisations are generalised associahedra associated to some cluster algebra of finite type. N. Reading and D. Speyer conjectured a linear isomorphism between Cambrian fans and g-vector fans associated to cluster algebras of finite type with acyclic initial seed introduced by S. Fomin and A. Zelevinsky, [6]. They proved their conjecture up to an assumption of another conjecture of [6]. In 2008, S.-W. Yang and A. Zelevinsky gave an alternative proof of the conjecture of Reading and Speyer, [20]. We remark in this context that the results of Section 2 and 3 of [11] can be read along these lines: the computations of z_I and y_I for fixed I and varying c involve sums over different choices of $\tilde{z}^c_{R_\delta}$ where the diagonals δ that have to be considered depend on c. Moreover, the values for $\tilde{z}^c_{R_\delta}$ that occur in these sums should be tight for the polytope but can be choosen within a large class of possible values as described for example in [9], not just the specific value chosen here in Section 2. The formula of Theorem 4 of this manuscript could be rewritten in this sense by introducing extra parameters. From this point of view, we suggest that combinatorial properties of the g-vector fan for cluster algebras of finite type A with respect to an acyclic initial seed are reflected by the Minkowski decompositions studied in [11] and in this manuscript.

Some instances of As^c_{n-1} have been studied earlier. For example, J.-L. Loday computes vertex coordinates from planar binary trees, [12]. This generalises $\mathsf{As}^{c_1}_2$ studied in Section 3 to higher dimensions. G. Rote, F. Santos, and I. Streinu relate associahedra to one-dimensional point configurations, [17]. Both realisations are affinely equivalent to As^c_{n-1} if $\mathsf{U}_c = \varnothing$ or $\mathsf{U}_c = [n] \setminus \{1, n\}$. Moreover, Rote et.al. point out that a realisation of F. Chapoton, S. Fomin, and A. Zelevinsky, [4], is not affinely equivalent to their realisation. But in fact, it is affinely equivalent to some As^c_{n-1}, i.e. $\mathsf{U}_c = \{2\}$ or $\mathsf{U}_c = \{3\}$ for $n = 4$. F. Santos and V. Pilaud recently constructed a family of polytopes called *brick polytopes* that are related to multitriangulations, [13]. As a special case, they obtain translates of the associahedra As^c_{n-1} studied in this paper. They describe brick polytopes as Minkowski sums of brick polytopes and in particular, they achieve a Minkowski decomposition different from ours. The precise relation of these two decompositions is not clear at the time of writing.

2 The associahedra As^c_{n-1}

Associahedra form a class of combinatorially equivalent simple polytopes and can be realised as generalised permutahedra. They are often defined by specifying their 1-skeleton or graph. A theorem of G. Kalai, [10], implies that the face lattice of an $(n-1)$-dimensional associahedron As_{n-1} is completely determined by this graph. Now, the graph of an associahedron is isomorphic to a graph with all triangulations (without new vertices) of a convex and plane $(n+2)$-gon Q as vertex set and all pairs of distinct triangulations that differ in precisely one proper diagonal[1] as edge set. Alternatively, the edges of As_{n-1} are in bijection with the set of triangulations with one proper diagonal removed. Similarly, k-faces of As_{n-1} are in bijection to triangulations of Q with k proper diagonals deleted. In particular, the facets of As_{n-1} are in bijection with proper diagonals of Q. J.-L. Loday published a beautiful algorithm to obtain explicit vertex coordinates for associahedra from planar binary trees, [12]. This algorithm was generalised by C. Hohlweg and C. Lange and explicitly describes realisations of As_{n-1} as generalised permutahedra that depend on combinatorics induced by the choice of a Coxeter element c of the symmetric group Σ_n on n elements, [8]. A Coxeter element is a permutation obtained by multiplying the generators of S_n in some order.

We now outline the construction of [8] and avoid to use Coxeter elements explicitly. Nevertheless, we use them to distinguish different realisations in our notation. The choice of a Coxeter element c corresponds to a partition of $[n]$ into a *down set* D_c and an *up set* U_c:

$$\mathsf{D}_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\}$$

and

$$\mathsf{U}_c = \{u_1 < u_2 < \cdots < u_m\}.$$

This partition induces a labeling of the vertices of Q with label set $[n+1]_0 := [n+1] \cup \{0\}$ as follows. Pick

[1] A proper diagonal is a line segment connecting a pair of vertices of Q whose relative interior is contained in the interior of Q. A non-proper diagonal is a diagonal that connects vertices adjacent in ∂Q and a degenerate diagonal is a diagonal where the end-points are equal.

two vertices of Q which are the end-points of a path of $\ell+2$ vertices on the boundary of Q, label the vertices of this path counter-clockwise increasing using the label set $\overline{\mathsf{D}}_c := \mathsf{D}_c \cup \{0, n+1\}$ and label the remaining path clockwise increasing using the label set U_c. Without loss of generality, we shall always assume that the label set D_c is to the right of the diagonal $\{0, n+1\}$ oriented from 0 to $n+1$, examples are given in Section 3. We derive values z_I for some subsets $I \subset [n]$ obtained from this labeled $(n+2)$-gon Q using proper diagonals of Q as follows. Orient each proper diagonal δ from the smaller to the larger labeled end-point of δ, associate to δ the set R_δ that consists of all labels on the strict right-hand side of δ, and replace the elements 0 and $n+1$ by the smaller respectively larger label of the end-points contained in U_c if possible. For each proper diagonal δ we have $R_\delta \subseteq [n]$ but obviously not every subset of $[n]$ is of this type if $n > 2$. We set

$$\tilde{z}_I^c := \begin{cases} \frac{|I|(|I|+1)}{2} & \text{if } I = R_\delta,\ \delta \text{ proper diagonal,} \\ -\infty & \text{else.} \end{cases}$$

In [8] it is shown that $P_n(\{\tilde{z}_I^c\})$ is in fact an associahedron of dimension $n-1$ realised in $\mathbb{R}^n$ for every choice of c and the inequalities that correspond to finite values $\tilde{z}_I^c$ are precisely the non-redundant facet-defining inequalities of As_{n-1}^c. This ends the summary of results found in [8].

To compute the coefficients of the Minkowski decomposition of As_{n-1}^c according to Proposition 1, we have to find tight values for z_I that correspond to all inequalities (redundant and non-redundant) first. Fortunately enough, this is not necessary. As outlined by the author in an extended abstract, [11], it suffices to know the finite values of $\tilde{z}_I^c$ defined above. To state and prove Theorem 4, we have to review some facts from [11] and start with two key definitions given there.

Suppose from now on that $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ is a partition of $[n]$ induced by a Coxeter element c with

$$\mathsf{D}_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\}$$

and

$$\mathsf{U}_c = \{u_1 < u_2 < \cdots < u_m\}.$$

Definition 1 (up and down intervals)

(a) A set $S \subseteq [n]$ is a non-empty interval of $[n]$ if $S = \{r, r+1, \cdots, s\}$ for some $0 < r \le s < n$. We write S as closed interval $[r, s]$ (end-points included) or as open interval $(r-1, s+1)$ (end-points not included). An empty interval is an open interval $(k, k+1)$ for some $1 \le k < n$.

(b) A non-empty open down interval is a set $S \subseteq \mathsf{D}_c$ such that $S = \{d_r < d_{r+1} < \cdots < d_s\}$ for some $1 \le r \le s \le \ell$. We write S as open down interval $(d_{r-1}, d_{s+1})_{\mathsf{D}_c}$ where we allow $d_{r-1} = 0$ and $d_{s+1} = n+1$, i.e. $d_{r-1}, d_{s+1} \in \overline{\mathsf{D}}_c$. For $1 \le r \le \ell - 1$, we have the empty down interval $(d_r, d_{r+1})_{\mathsf{D}_c}$.

(c) A closed up interval is a non-empty set $S \subseteq \mathsf{U}_c$ such that $S = \{u_r < u_{r+1} < \cdots < u_s\}$ for some $1 \le r \le s \le m$. We write $[u_r, u_s]_{\mathsf{U}_c}$.

We emphasize that up intervals are always non-empty, while down intervals may be empty. Moreover, it turns out to be convenient to distinguish the empty down intervals $(d_r, d_{r+1})_{\mathsf{D}_c}$ and $(d_s, d_{s+1})_{\mathsf{D}_c}$ if $r \ne s$ although they are equal as sets.

Definition 2 (up & down interval decomposition) *Let I be a non-empty subset of $[n]$.*

(a) An up and down interval decomposition of type (v, w) of I is a partition of I into disjoint up and down intervals $I_1^{\mathsf{U}}, \cdots, I_w^{\mathsf{U}}$ and $I_1^{\mathsf{D}}, \cdots, I_v^{\mathsf{D}}$ obtained by the following procedure.

1. Suppose there are $\tilde{v}$ non-empty inclusion maximal down intervals contained in I that we denote by $\tilde{I}_k^{\mathsf{D}} = (\tilde{a}_k, \tilde{b}_k)_{\mathsf{D}_c}$, $1 \le k \le \tilde{v}$, with $\tilde{b}_k \le \tilde{a}_{k+1}$ for $1 \le k < \tilde{v}$. Let $E_i^{\mathsf{D}} = (d_{r_i}, d_{r_i+1})_{\mathsf{D}_c}$ denote all empty down intervals with $\tilde{b}_k \le d_{r_i} < d_{r_i+1} \le \tilde{a}_{k+1}$ for $0 \le k \le \tilde{v}$, $\tilde{b}_0 = 0$, and $\tilde{a}_{\tilde{v}+1} = n+1$. Denote the open intervals $(\tilde{a}_i, \tilde{b}_i)$ and (d_{r_i}, d_{r_i+1}) of $[n]$ by $\tilde{I}_i$ and E_i respectively.

2. Consider all up intervals of I which are contained in (and inclusion maximal within) some interval $\tilde{I}_i$ or E_i obtained in Step 1 and denote these up intervals by

$$I_1^{\mathsf{U}} = [\alpha_1, \beta_1]_{\mathsf{U}_c}, \cdots, I_w^{\mathsf{U}} = [\alpha_w, \beta_w]_{\mathsf{U}_c}.$$

We assume $\alpha_i \le \beta_i < \alpha_{i+1}$.

3. A down interval $I_i^{\mathsf{D}} = (a_i, b_i)_{\mathsf{D}_c}$, $1 \le i \le w$, is a down interval obtained in Step 1 that is either a non-empty down interval $\tilde{I}_k^{\mathsf{D}}$ or an empty down interval E_k^{D} with the additional property that there is some up interval I_j^{U} obtained in Step 2 such that $I_j^{\mathsf{U}} \subseteq E_k$. Without loss of generality, we assume $b_i \le a_{i+1}$ for $1 \le i < w$.

(b) An up and down interval decomposition of type $(1, w)$ is called nested. A nested component of I is an inclusion-maximal subset J of I such that the up and down decomposition of J is nested.

The up and down interval decomposition of $I \subseteq [n]$ enables us to compute tight values $\tilde{z}_I^c$ of As_{n-1}^c for all I using only $\tilde{z}_I^c$ that correspond to non-redundant inequalities. These values can be substituted in the formula for y_I of Proposition 1 and the formula can be simplified significantly. Before we state the resulting theorem, it makes sense to extend our notion of R_δ and $\tilde{z}_{R_\delta}^c$ to non-proper and degenerate diagonals δ.

For a diagonal $\delta = \{x, y\}$ that is not proper, we set

$$R_\delta := \begin{cases} \varnothing & \text{if } x, y \in \overline{\mathsf{D}}_c \\ [n] & \text{otherwise,} \end{cases}$$

and

$$\tilde{z}^c_{R_\delta} := \begin{cases} 0 & \text{if } R_\delta = \varnothing \\ \frac{n(n+1)}{2} & \text{if } R_\delta = [n]. \end{cases}$$

Let $I \subseteq [n]$ be a non-empty subset with up and down interval decomposition of type (v, k). If I has a nested up and down interval decomposition, then, in particular, $v = 1$ and

$$I = (a, b)_{\mathsf{D}_c} \cup \bigcup_{i=1}^{k} [\alpha_i, \beta_i]_{\mathsf{U}_c}$$

with $\alpha_k < \beta_k \leq \alpha_{k+1}$ as before. In this situation, we denote the smallest (respectively largest) element of I by γ (respectively Γ) and consider the diagonals

$$\delta_1 := \{a, b\}, \qquad \delta_2 := \{a, \Gamma\},$$
$$\delta_3 := \{\gamma, b\}, \text{ and } \qquad \delta_4 := \{\gamma, \Gamma\}.$$

We can now state the main result of [11] which we use to prove Theorem 4.

Theorem 2 ([11, Theorem 3.1])
Let I be a non-empty subset of $[n]$ with a nested up and down interval decomposition of type $(1, k)$. Then

$$y_I = (-1)^{|I \setminus R_{\delta_1}|} \left(z^c_{R_{\delta_1}} - z^c_{R_{\delta_2}} - z^c_{R_{\delta_3}} + z^c_{R_{\delta_4}} \right).$$

Corollary 3 ([11, Corollary 3.2])
Let I be a non-empty subset of $[n]$ with an up and down interval decomposition of type (v, k) and $v > 1$. Then $y_I = 0$.

3 Main theorem and examples

We continue to use the notation introduced in the previous section. Moreover, we need the notion of signed lengths K_γ and K_Γ for sets I with interval decomposition of type $(1, k)$ that is needed in Step 2. (b) of Theorem 4. They denote integers and have the following meaning: $|K_\Gamma|$ is the length, i.e. the number of edges, of the path in ∂Q connecting b and Γ that does not use the vertex labeled a and K_Γ is negative if and only if $\Gamma \in (a, b)_{\mathsf{D}}$. Similarly, $|K_\gamma|$ is the length of the path connecting a and γ that does not use the vertex labeled b and K_γ is negative if and only if $\gamma \in (a, b)_{\mathsf{D}}$.

Theorem 4 *Let Q be the $(n+2)$-gon labeled according to the construction of As^c_{n-1} and $I \subseteq [n]$ be non-empty. To compute y_I perform the following two steps:*

1. *Determine the type (v, w) of the up and down interval decomposition of I.*
2. *(a) If $v > 1$ then $y_I = 0$.*
   (b) If $v = 1$ then
   $$y_I = (-1)^{|I \setminus (a,b)_{\mathsf{D}}|} \left(K_\gamma K_\Gamma - (n+1) \right)$$
   if $|I| = 1$ and $I \subseteq \mathsf{U}$, while
   $$y_I = (-1)^{|I \setminus (a,b)_{\mathsf{D}}|} K_\gamma K_\Gamma$$
   otherwise.

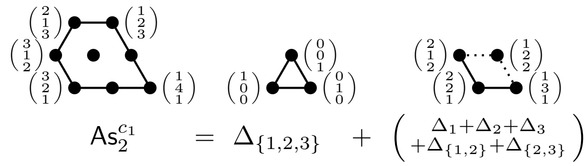


Figure 2: The Minkowski decomposition of the 2-dimensional associahedron $\mathsf{As}^{c_1}_2$ into faces of the standard simplex is in fact a Minkowski sum.

Theorem 4 is the third to relate combinatorics of labeled n-gons to different aspects of realisations of associahedra. Firstly, the coordinates of the vertices can be extracted, [12, 8]. Secondly, the facet normals and the right-hand sides for their inequalities can be read off, [8]. Thirdly, the coefficients of a Minkowski decomposition are obtained according to Theorem 4.

Before giving the proof, we give an example of two 2-dimensional associahedra $\mathsf{As}^{c_1}_2$ and $\mathsf{As}^{c_2}_2$. The first example $\mathsf{As}^{c_1}_2$ corresponds to $\mathsf{D}_{c_1} = [n]$ and $\mathsf{U}_{c_1} = \varnothing$. Minkowski decompositions of $\mathsf{As}^{c_1}_2$ and its higher dimensional analogues were already studied earlier as mentioned by A. Postnikov and it is known that $y_I \in \{0, 1\}$, so these polytopes are actually a Minkowski sum of faces of the standard simplex. We have

$$\mathsf{As}^{c_1}_2 = \Delta_{\{1\}} + \Delta_{\{2\}} + \Delta_{\{3\}} + \Delta_{\{1,2\}} + \Delta_{\{2,3\}} + \Delta_{\{1,2,3\}},$$

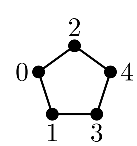
see Figure 2. Although $\mathsf{As}^{c_2}_2$ is isometric to $\mathsf{As}^{c_1}_2$, it does not decompose into a Minkowski sum of dilated faces of a standard simplex but into a Minkowski sum and difference of dilated faces of the standard simplex:

$$\mathsf{As}^{c_2}_2 = \left(\begin{smallmatrix} \Delta_{\{1\}}+\Delta_{\{3\}}+2\cdot\Delta_{\{1,2\}} \\ +\Delta_{\{1,3\}}+2\cdot\Delta_{\{2,3\}} \end{smallmatrix} \right) - \Delta_{\{1,2,3\}},$$

see Figure 3. The up and down sets in this situation are

$$\mathsf{U}_c = \{2\} \quad \text{and} \quad \mathsf{D}_c = \{1, 3\},$$

so we obtain the following labeled pentagon Q:



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We now compute the coefficients $y_{\{2\}}$, $y_{\{1,2\}}$, and $y_{\{1,2,3\}}$ in order to demonstrate Theorem 4.

The up and down interval decompositions for $\{2\}$, $\{1, 2\}$, and $\{1, 2, 3\}$ are of type $(1, 1)$:

$$\{2\} = (1, 3)_D \sqcup [2, 2]_{\mathsf{U}},$$
$$\{1, 2\} = (0, 3)_D \sqcup [2, 2]_{\mathsf{U}},$$
$$\{1, 2, 3\} = (0, 4)_D \sqcup [2, 2]_{\mathsf{U}}.$$

Hence we obtain the following table:

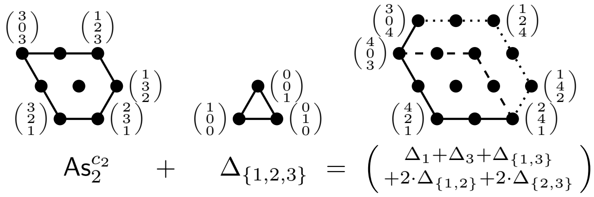

Figure 3: The Minkowski decomposition of the 2-dimensional associahedron $\mathsf{As}_2^{c_2}$ into dilated faces of the standard simplex.

I	a	b	γ	Γ	K_γ	K_Γ	$\|I \setminus (a,b)_{\mathsf{D}}\|$
{2}	1	3	2	2	2	2	1
{1,2}	0	3	1	2	−1	2	1
{1,2,3}	0	4	1	3	−1	−1	1

Since $n = 3$ in this example, we compute

$$\begin{aligned} y_{\{2\}} &= (-1)^1(2\cdot 2 - (3+1)) = 0, \\ y_{\{1,2\}} &= (-1)^1 \cdot (-1) \cdot 2 = 2, \\ y_{\{1,2,3\}} &= (-1)^1 \cdot (-1) \cdot (-1) = -1. \end{aligned}$$

4 Proof of the main theorem

The strategy of the proof is clear: Suppose $I \subseteq [n]$ is non-empty, we compute the up and down interval decomposition (Step 1. of Theorem 4) and then reinterpret Theorem 2 and Corollary 3 in terms of K_γ and K_Γ. If the up and down decomposition of I is of type (v, w) with $v \geq 2$ then the claim of Step 2. (a) follows immediately from Corollary 3. We therefore assume that I has an up and down interval decomposition of type $(1, k)$, the associated down interval is $(a, b)_D$ and the minimal and maximal elements of I are γ and Γ. We also use the notation of δ_i, $1 \leq i \leq 4$, from Section 2 and define

$$\widetilde{K}_\Gamma := |R_{\delta_2}| - |R_{\delta_1}| \quad \text{as well as} \quad \widetilde{K}_\gamma := |R_{\delta_3}| - |R_{\delta_1}|.$$

A simple case-by-case analysis shows

1. $\widetilde{K}_\gamma > 0$ if and only if $\gamma \in \mathsf{U}_c$.
2. $\widetilde{K}_\Gamma > 0$ if and only if $\Gamma \in \mathsf{U}_c$.
3. $\widetilde{K}_\gamma = -1$ if and only if $\gamma \in \mathsf{D}_c$.
4. $\widetilde{K}_\Gamma = -1$ if and only if $\Gamma \in \mathsf{D}_c$.

as well as $K_\Gamma = \widetilde{K}_\Gamma$ and $K_\gamma = \widetilde{K}_\gamma$. We additionally define $K := |R_{\delta_1}|$ and a direct computation allows to express $z^c_{R_{\delta_i}}$, $1 \leq i \leq 3$, in terms of K, K_Γ, and K_γ:

$$\begin{aligned} z^c_{R_{\delta_1}} &= \frac{K(K+1)}{2}, \\ z^c_{R_{\delta_2}} &= \frac{(K+K_\Gamma)(K+K_\Gamma+1)}{2}, \\ \text{and } z^c_{R_{\delta_3}} &= \frac{(K+K_\gamma)(K+K_\gamma+1)}{2}. \end{aligned}$$

Another direct computation yields

$$K_\Gamma K_\gamma = z_{R_{\delta_1}} - z_{R_{\delta_2}} - z_{R_{\delta_3}} + \frac{(K+K_\Gamma+K_\gamma)(K+K_\Gamma+K_\gamma+1)}{2}.$$

To express $\tilde{z}^c_{R_{\delta_4}}$ in terms of K, K_Γ, and K_γ, we observe

$$\frac{(K+K_\Gamma+K_\gamma)(K+K_\Gamma+K_\gamma+1)}{2} = \begin{cases} \frac{|R_{\delta_4}|(|R_{\delta_4}|+1)}{2} & \text{if } I \neq \{u_s\}, \\ \frac{|R_{\delta_4}|(|R_{\delta_4}|+1)}{2} + (n+1) & \text{if } I = \{u_s\}, \end{cases}$$

and obtain

$$z^c_{R_{\delta_4}} = \frac{(K+K_\Gamma+K_\gamma)(K+K_\Gamma+K_\gamma+1)}{2}$$

if $I \neq \{u_s\}$, and

$$z^c_{R_{\delta_4}} = \frac{(K+K_\Gamma+K_\gamma)(K+K_\Gamma+K_\gamma+1)}{2} - (n+1)$$

if $I = \{u_s\}$. The claim follows now from Theorem 2.

5 A Remark on Cyclohedra

Cyclohedra are also known as Bott-Taubes polytopes or type B generalised associahedra, [3, 4, 19]. They can be realised using some As^c_{n-1} by intersection with *type B hyperplanes* $x_i + x_{2n+1-i} = 2n+1$, $1 \leq i < n$. We refer to [8] for details. A 2-dimensional cyclohedron Cy^c_2 obtained from some As^c_3 by intersection with $x_1 + x_4 = 5$ is shown in Figure 4. Tight right-hand sides for Cy^c_2 are the right-hand sides of As^c_2 except $z_{\{1,4\}}$ and $z_{\{2,3\}}$ whose tight value is 5 instead of 2. The inequalities $x_1 + x_4 \geq 2$ and $x_2 + x_3 \geq 2$ are redundant for As^c_2 and altering the level sets for these inequalities from 2 (for As^c_2) to 5 (for Cy^c_2) means that we move past the four vertices A, B, C, and D. As explained in [15], this

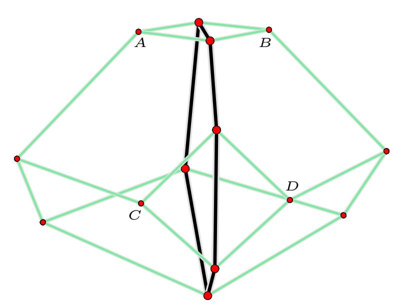

Figure 4: A 2-dimensional cyclohedron Cy^c_2 (indicated in black) obtained from an associahedron As^c_3 by intersection with type B hyperplanes.

implies that Cy_2^c is not in the deformation cone of the classical permutahdron. Applying Proposition 1 to the function z_I on the boolean lattice for Cy_2^c, we compute the Möbius inverse y_I. We obtain

$$\mathsf{Cy}_2^c + \left({\Delta_2+4\Delta_{123} \atop +3\Delta_{124}+2\Delta_{134}+\Delta_{234}} \right) = \left({\Delta_1+\Delta_3+\Delta_4+3\Delta_{12}+\Delta_{13} \atop +3\Delta_{14}+5\Delta_{23}+\Delta_{34}+5\Delta_{1234}} \right)$$

if Proposition 1 were true for polytopes $P_n(\{z_I\})$ not contained in the deformation cone of the classical permutahedron. One way to see that this equation does not hold is to compute the number of vertices of the polytope on the left-hand side (27 vertices) and on the right-hand side (20 vertices) using `polymake`, [7].

6 Concluding remarks

There are some questions related to the coefficients y_I. Firstly, how do Minkowski decompositions of generalised associahedra obtained in [9] look like and how can we compute them if they exit? In particular, how to decompose the cyclohedron of [8]?

Secondly, the computation of the Minkowski decomposition depends on a good understanding of computational aspects of Möbius inversions. Efficient computations of Möbius functions on lattices were studied by A. Blass and B. Sagan, [2]. Does this extend somehow to Möbius inversion? Moreover, Möbius inversions are often used in proofs but because of the computational complexity rarely used in computations. Is there some theory to compute the Möbius inverse more efficiently?

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