Illumination problems on translation surfaces with planar infinities

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Abstract

In the current article we discuss an illumination problem proposed by Urrutia and Zaks. The focus is on configurations of finitely many two-sided mirrors in the plane together with a source of light placed at an arbitrary point. In this setting, we study the regions unilluminated by the source. In the case of rational-π angles between the mirrors, a planar configuration gives rise to a surface with a translation structure and a number of planar infinities. We show that on a surface of this type with at least two infinities, one can find plenty of unilluminated regions isometric to unbounded planar sectors. In addition, we establish that the non-bijectivity of a certain circle map implies the existence of unbounded dark sectors for rational planar mirror configurations illuminated by a light-source.

1 Introduction

Consider a planar domain with a light reflecting boundary. Place a source of light at a point inside the domain. Assume that the source emits rays in all directions. Each ray follows a straight line and whenever it reaches the boundary it is reflected according to the rule that the angle of incidence equals the angle of reflection. A point from the domain is considered *illuminated* by the source whenever there is a ray that reaches the point either directly or after a series of reflections. In this setting, one can ask the following questions, also known as *illumination problems*.

Question 1 *If we place the source of light at any point in the domain, will all of the domain be illuminated? If not, what could be said about the non-illuminated regions?*

Question 2 *Is there a point from which the light source can illuminate the entire domain?*

These problems are often attributed to E. Straus who posed them sometime in the early fifties and first published by V. Klee in 1969 [5]. Some famous examples and interesting results are Penrose's room [1], Tokarsky's example [5] as well as the article [3] by Hubert, Schmoll and Troubetzkoy on illumination on Veech surfaces.

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In 1991, J. Urrutia and J. Zaks proposed the following problem [6]. Assume we are given a finite number of disjoint compact line segments in the plane each representing a mirror that reflects light on both sides (a two-sided mirror). Let p_0 be any point on the plane not incident to any of the segments. Then, the complement of the set of mirrors is an unbounded domain with light-reflecting boundary and if we place a source of light S at p_0 we can pose questions 1 and 2. Figure 1a depicts an example of a two-sided mirror configuration with a light emitting source S. The convex hull of the mirrors is a polygon. If S is in the convex hull, one can construct a triangle P unilluminated by S, like the shaded one on figure 1b. To do that, it is sufficient for a mirror segment to be an edge of the convex hull.

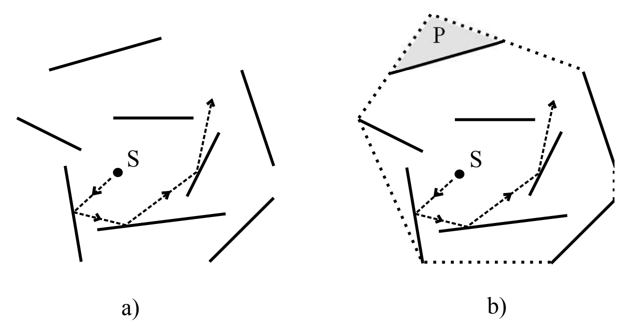

Figure 1:

In this paper we are interested in finite two-sided mirror configurations with the following property: any pair of lines determined by the mirror segments are either parallel or intersect at an angle which is a rational multiple of π. We will call such a configuration a *rational mirror configuration* and the domain obtained as a complement of the mirrors will be called *rational mirror domain*. For those, we will find conditions that will guarantee the existence of unbounded unilluminated sectors in the plane (see definition 2).

A rational mirror domain can be "unfolded" into a surface that carries a flat metric with conical singularities and trivial holonomy group (see section 3 or [2, 4]). This means that the surface has a special atlas, called a *translation atlas*, with the property that *away from the cone points, the transition maps between two charts from the atlas are Euclidean translations* (section 3 or [2, 4]). As a result, the piecewise linear trajectory of a light ray in the original domain becomes a smooth geodesic on the flat surface. Thus, one can think of a light source placed at a nonsingular point on the surface, emitting geodesic rays in all directions. Any other point is considered *illuminated* if there is a smooth geodesic connecting the source to the point. In this way, one can ask questions 1 and 2 for the surface. Notice that there are regions on it isometric to complements of compact sets in the plane. We will call a surface with such a geometry *a translation surface with planar infinities*.

A translation surface with planar infinities gives rise to a pair (X, ω) where X is a closed surface with a complex structure and ω is a meromorphic differential on X with only double poles and zero residues. The zeroes of ω are the cone points of the flat structure [2, 4], and around each pole the surface looks like the complement of a compact set in the plane. The converse is also true. A pair (X, ω) of a closed Riemann surface and a meromorphic differential with only double poles and zero residues induces a translation structure on X with planar infinities. We have provided more details, definitions and constructions in section 3. For a good introduction to the theory of polygonal billiards and *translation surfaces*, we recommend [2] and [4].

Definition 1 *The pair (X, ω) is called a translation surface with planar infinities whenever the following conditions hold:*

(1) *X is a closed surface with a complex structure;*

(2) *ω is a meromorphic differential on X;*

(3) *Every pole of ω is of order exactly 2 and the residue at that pole is zero. We will refer to the poles of ω as planar infinities.*

In this study we would like to show non-illumination of a special type of domains both on a translation surface with planar infinities and in the plane.

Definition 2 a) *Let l_1 and l_2 be two half-lines in the plane both starting form a point p_0 and going to infinity. Let θ be the angle between l_1 and l_2 at the vertex p_0, measured counterclockwise from l_1 to l_2. Then, the open region C bounded by l_1 and l_2, whose internal angle at p_0 is θ, is called an infinite sector of angle θ (see figure 2a).*

b) *An open subdomain C of a translation surface with planar infinities (X, ω) is called an infinite sector of angle θ whenever there exists a chart from the translation atlas of (X, ω) that maps C isometrically to a planar infinite sector of angle θ like the one defined in point a.*

On any translation surface (X, ω) one can always find an orientable foliation $\mathcal{F}_\omega$ with singularities, whose leaves are geodesics. Indeed, let us foliate the Euclidean plane into horizontal straight lines, oriented as usual from left to right. Since each transition map between two charts is a Euclidean translation, it sends horizontal lines to horizontal lines (line orientation preserved). Thus, pulling back onto the surface the planar horizontal foliation from all translation charts defines globally the desired foliation $\mathcal{F}_\omega$. Moreover, the singularities of $\mathcal{F}_\omega$ are the cone points of the surface (X, ω), i.e. the zeroes of the differential ω. We call $\mathcal{F}_\omega$ the *horizontal foliation* of the surface and its leaves - the *horizontal geodesics* of the surface. At each non-singular point p_0 of (X, ω) the oriented horizontal geodesic $l_{p_0}(0)$ from $\mathcal{F}_\omega$ defines a *positive horizontal direction* at p_0. The counterclockwise angle α between $l_{p_0}(0)$ and an arbitrary oriented geodesic $l_{p_0}(\alpha)$ through p_0 is called the *direction* of $l_{p_0}(\alpha)$ at p_0 (see figure 2b). From now on, $l_{p_0}(\alpha)$

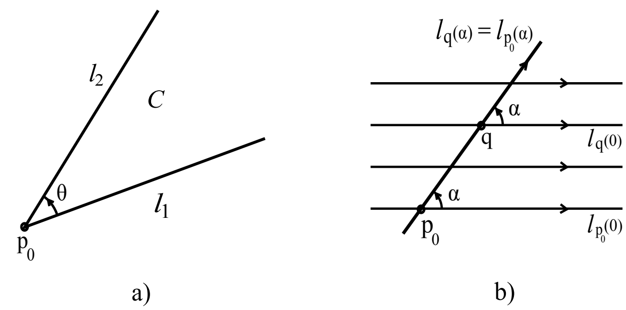

Figure 2:

denotes the geodesic ray on (X, ω) starting from $p_0 \in X$ and going in the direction of angle α. It is important to emphasize that, since we are working with a translation surface, the intersection of the geodesic $l_{p_0}(\alpha)$ with any other horizontal geodesic $l_q(0)$ will always form the same angle α, as shown locally on figure 2b. In other words, just like in the plane, a geodesic on (X, ω) does not changes its angle with respect to the horizontal direction. Since a direction at any non-singular point $p \in X$ is defined as an angle $\alpha \in \mathbb{R} \mod 2\pi$, we can identify the set of all directions at p with the unit circle $\mathrm{S}^1 = \{z \in \mathbb{C} \, : \, |z| = 1\}$. The point $1 \in \mathrm{S}^1$ gives the horizontal direction $\alpha = 0$.

2 Results

It is natural to ask questions about the behavior of the geodesics on a surface. The first question we will address is the following. *On a translation surface with planar infinities, where do most geodesics emanating from a nonsingular point go?* As it turns out, almost all of them fall onto the poles of the surface. Same is true for any rational mirror configuration in the plane.

Theorem 1 *The following two statements are true:*

(1) *Let (X, ω) be a translation surface with planar infinities and let $p_0 \in X$ be non-singular. Then the set of all directions $\alpha \in \mathrm{S}^1$, for which the geodesic passing*

through p_0 in direction α goes to one of the poles of ω, is open and dense in the circle S^1;

(2) *Assume we are given a rational mirror configuration in the plane and let p_0 be a point not lying on any of the mirrors. Then the set of all directions $\alpha \in S^1$, for which the piece-wise linear reflected trajectory starting from p_0 in direction α goes to infinity, is open and dense in the circle S^1.*

The next result establishes the existence of infinite unilluminated sectors and large unbounded regions on translation surfaces with more than one planar infinity.

Theorem 2 *Let (X, ω) be a translation surface with at least two planar infinities. Then, for any point p_0 on $X \setminus (zeroes(\omega) \cup poles(\omega))$ there exists an infinite sector C on (X, ω) unilluminated by p_0, i.e. for any point $p \in C$ there is no smooth geodesic on (X, ω) that connects p_0 to p. Moreover, there exists a region on (X, ω) consisting of unilluminated, non-overlapping infinite sectors of total angle $2\pi(k-1)$, where k is the number of poles of ω.*

The main ideas used in the proof of theorem 2 can be adjusted to the study of illumination problems for rational mirror configurations in the plane. For instance, an interesting question put in an every day language, is the following. *How big of an object can be hidden from a stationary observer in a rational mirror domain? Can we hide a car? A whole parking lot of cars?* Precisely speaking, we would like to find a basic condition that will ensure the existence of an infinite unilluminated sector for a light source placed at a point inside a rational mirror domain.

Let D be a rational mirror domain and let $p_0 \in D$. Draw a large enough circle K, so that its interior contains the mirrors from the configuration and the light source at the point p_0. Denote by U_{p_0} the open dense set of all directions which go to infinity, provided by theorem 1. For an angle $\alpha \in U_{p_0} \subset S^1$ follow the straight line $l_{p_0}(\alpha)$ starting form p_0 in direction of α. Whenever

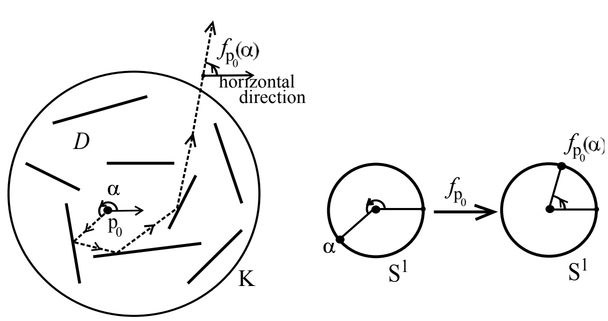

Figure 3:

the line reaches a mirror it is reflected, changing its direction. In this way, a piecewise linear trajectory is formed, which at some point leaves the disc bounded by K never to come back to it. Denote by $f_{p_0}(\alpha)$ the angle between the horizontal direction of $\mathbb{C}$ and the portion of the trajectory that is outside the circle K. As a result, we obtain a map $f_{p_0} : U_{p_0} \longrightarrow S^1$. For a picture of the construction of f_{p_0} see figure 3. The map f_{p_0} is defined almost everywhere on the unit circle. In fact, its domain U_{p_0} is open and dense in S^1. Moreover, f_{p_0} is a rotation when restricted to any connected component of U_{p_0}. Our hope is that finding ways to study the combinatorial properties of f_{p_0} may facilitate the search for unbounded unilluminated sectors in rational mirror domains.

Theorem 3 *Assume we are given a rational mirror configuration. For an arbitrary point p_0 not on any of the mirrors, consider the circle map f_{p_0} (see figure 3). If f_{p_0} is not injective, then there exists an infinite sector in the plane unilluminated by p_0.*

3 Translation surfaces.

In the current section we discuss translation surfaces and show how to construct one from a rational mirror configuration. To illustrate the idea better, we apply the procedure to an example.

Various descriptions. A translation surface is a closed surface X with a finite set of points $\Sigma \subset X$, called singularities, and a cover of $X \setminus \Sigma$ by open charts $\{(W_a, \varphi_a) \mid W_a \subseteq X \setminus \Sigma,\ \varphi_a : W_a \to \mathbb{C}\}$ having the property that whenever $W_a \cap W_b \neq \varnothing$ the transition map between the two charts (W_a, φ_a) and (W_b, φ_b) is a Euclidean translation, i.e. $z_b = \varphi_b^{-1} \circ \varphi_a(z_a) = z_a + c$. In our study, Σ partitions into two subsets Σ_0 and Σ_∞. Each point from Σ_0 has a cone angle of $2\pi N$, where N is a positive integer. Each point p_∞ form Σ_∞ has an open neighborhood $W' \subset X$ with a map $\varphi_\infty : W' \setminus \{p_\infty\} \to \mathbb{C}$ such that $(W' \setminus \{p_\infty\}, \varphi_\infty)$ is a translation chart from the atlas and the set $\mathbb{C} \setminus \varphi_\infty(W' \setminus \{p_\infty\})$ is compact. Thus, the collection Σ_∞ contains all planar infinities on the surface.

Since translations are holomorphic maps, the translation atlas induces a complex structure on X (for details see [2] and [4]). Moreover, the differential dz_a in each $\varphi(W_a) \subset \mathbb{C}$ can be pulled back as a holomorphic differential $\omega_a = \varphi_a^* dz_a$ in the corresponding W_a. But if $z_b = \varphi_b^{-1} \circ \varphi_a(z_a) = z_a + c$ then $dz_b = dz_a$. Hence, $\omega_a = \omega_b$ in any intersection $W_a \cap W_b \neq \varnothing$ which gives rise to a global holomorphic differential ω on $X \setminus \Sigma$. Moreover, ω extends to the singular set Σ so that Σ_0 becomes the set of zeroes of ω and Σ_∞ becomes the set of all poles of ω. The latter are all double and with residue 0. So we see that a translation surface with planar infinities induces a pair (X, ω) of a compact Riemann surface without boundary together with an appropriate meromorphic differential.

To recover the translation atlas from a pair (X,ω), one can cover $X \setminus (\text{zeroes}(\omega))$ with topological discs W_a. On each of them define the chart $\varphi_a(p) = \int_{p_a}^{p} \omega$, where $p_a \in W_a$ is fixed and p varies in W_a. As ω is either holomorphic or meromorphic with a double pole and residue 0 inside the topological disc W_a, the path of integration in $W_a \setminus \text{poles}(\omega)$ is arbitrary. If $W_a \cap W_b \neq \varnothing$ then $z_b = \int_{p_b}^{p} \omega = \int_{p_a}^{p} \omega + \int_{p_b}^{p_a} \omega = z_a + c$ for $p \in W_a \cap W_b$. Thus, we have obtained the desired translation atlas. As we can see, the description of a translation surface with planar infinities which we gave in the beginning of the current section is equivalent to definition 1.

The horizontal foliation $\mathcal{F}_\omega$ on X, mentioned in the introduction, is defined as follows. Let $\mathcal{F}_{\mathbb{C}}$ be the foliation of horizontal lines $\{z \in \mathbb{C} \,|\, \text{Im}(z) = s\}$, $s \in \mathbb{R}$ in $\mathbb{C}$ oriented from left to right (see figure 2b). Define the pulled-back local foliation $\mathcal{F}_a = \varphi_a^* \mathcal{F}_{\mathbb{C}}$ in each W_a. Observe that $\mathcal{F}_{\mathbb{C}}$ is invariant with respect to any translation, i.e. the translations map any horizontal line to a horizontal line. Hence, $\mathcal{F}_a = \mathcal{F}_b$ on each $W_a \cap W_b \neq \varnothing$. Thus, all local foliations fit together in a global foliation $\mathcal{F}_\omega$ on X with geodesic leaves and singularities Σ. The oriented leaves of $\mathcal{F}_\omega$ determine globally a horizontal direction on (X,ω). Since translations are Euclidean isometries, the Euclidean metric on $\mathbb{C}$ induces a Euclidean metric on $X \setminus \Sigma$. In this metric geodesics that do not go through singularities are isometric to straight lines in $\mathbb{C}$. The notion of a direction at a non-singular point $p \in X$ is as defined in the introduction. It is the counterclockwise angle between the horizontal leaf and an oriented geodesic both passing through p. Finally, an oriented geodesic always forms the same angle with any horizontal leaf it intersects, so it never self-intersects, except possibly to close up.

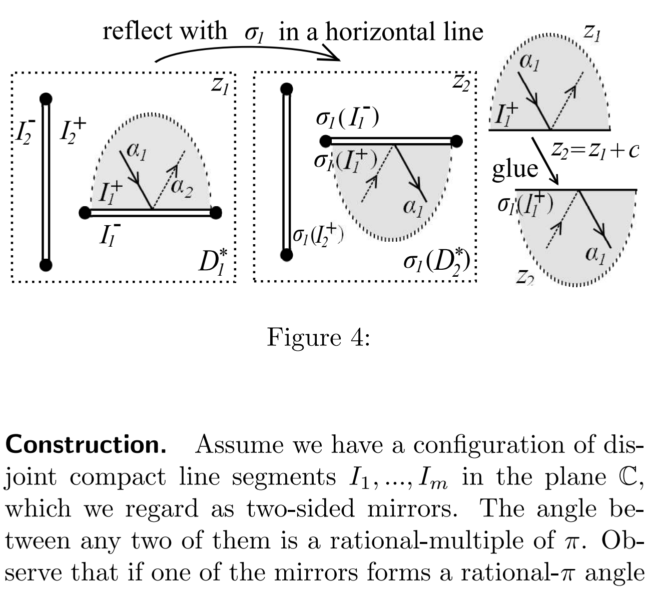

Figure 4:

Construction. Assume we have a configuration of disjoint compact line segments $I_1, ..., I_m$ in the plane $\mathbb{C}$, which we regard as two-sided mirrors. The angle between any two of them is a rational-multiple of π. Observe that if one of the mirrors forms a rational-π angle with the rest of the mirrors, then immediately follows that any pair of mirrors forms a rational-π angle. This is a consequence of the fact that in an Euclidean triangle the angles at the vertices sum up to π.

To understand better the construction that follows, one could have a simple toy-example in mind. Let us have two perpendicular mirrors I_1 and I_2 in the plane $\mathbb{C}$ like the ones depicted on figure 4.

Begin by slicing $\mathbb{C}$ along the segments $I_1, ..., I_n$ to obtain a closed slitted domain D^* in which every mirror segment I_k is doubled in order to obtain two parallel copies I_k^+ and I_k^- that form the boundary component of the surface D^* around the slit I_k. For an intuitive geometric picture of D^* in the case of the toy-example, look at figure 4. Then D^* is homeomorphic to a once-punctured sphere with n disjoint open discs removed, as shown on figure 5 for the case of two orthogonal mirrors. In particular, $\partial D^* = \sqcup_{k=1}^{n} (I_k^+ \cup I_k^-)$.

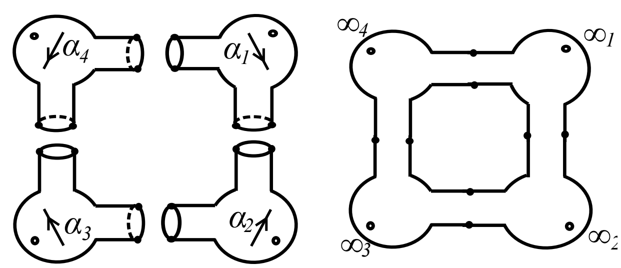

Figure 5:

For each segment I_k, fix the line $l_k \subset \mathbb{C}$ through $0 \in \mathbb{C}$ parallel to I_k. Denote by σ_k the reflection of $\mathbb{C}$ in l_k. The group G generated by all σ_k, $k = 1,..,n$ is a finite group. If α_1 is a generic direction in $\mathbb{C}$, then $G(\alpha_1) = \{g(\alpha_1) \,|\, g \in G\} = \{\alpha_1, ..., \alpha_m\}$ is an orbit of maximal length $m \leq n$. In our example $G \cong \mathbb{Z}_4$ and a generic orbit has 4 elements. Pick m copies D_j^* of D^* each with a choice of a direction α_j in it. If you prefer more formally, let $D_j^* = (D^*, \alpha_j)$. On figure 5, in the case of the toy-example, we can see a topological model of these four slitted planes with a choice of direction on each of them. We glue D_i^* to D_j^* if and only if there is a segment $I_k \subset \mathbb{C}$ whose corresponding reflection σ_k satisfies $\sigma_k(\alpha_i) = \alpha_j$. The gluing is done in the following way. Take D_i^* and $\sigma_k(D_j^*)$. Glue the edge $I_k^+ \subset D_i^*$ to the edge $\sigma_k(I_k^+) \subset \sigma_k(D_j^*)$ and the edge $I_k^- \subset D_i^*$ to the edge of $\sigma_k(I_k^-) \subset \sigma_k(D_j^*)$. On figure 4 of the toy-example, we have chosen $i = 1$ and $j = 2$. The upper edge $I_1^+ \subset D_1$ of the cut I_1 is glued to the lower edge $\sigma_1(I_1^+) \subset \sigma_1(D_2^*)$ of the cut $\sigma_1(I_1)$. Analogously, the lower edge I_1^- from D_1^* is glued to upper edge $\sigma(I_1^-)$ from $\sigma_1(D_2^*)$.

Both D_i^* and $\sigma_k(D_j^*)$ are naturally translation surfaces with piecewise geodesic boundaries, global coordinates z_i and z_j, and differentials dz_i and dz_j respectively. Segments I_k and $\sigma_k(I_k)$ are equal and parallel, hence the gluing map is a translation $z_j = z_i + c$ (see the gluing of the shaded pieces on figure 4). Therefore the resulting surface made out of D_i^* and $\sigma_k(D_j^*)$ has a translation structure. Moreover, $dz_j = dz_i$ along the gluing locus, so there is a well-defined holomorphic differential on the new surface which extends meromorphically to both of its infinity points.

Now, follow the described gluing procedure for all cuts on the pieces D_j^*, where $j = 1, .., m$. The final result is a closed Riemann surface X and a meromorphic differential ω with only double poles and zero residues, as well as simple zeroes with cone angle 4π. For the example of the two orthogonal mirrors, figure 5 illustrates how the four pieces $D_1^*, ..., D_4^*$ fit together to form a compact torus X with a complex structure and a meromorphic differential ω on X. There are eight simple zeroes of ω and four double poles. The zeroes are obtained from identifying pairs of black vertices on the segments I_k form figure 4. The cone angle at each zero is 4π and the residue at each pole is 0 as desired.

4 Proofs

Proof of theorem 1. From now on (X, ω) is an arbitrary translation surface with planar infinities and $p_0 \in X \setminus (\text{zeroes}(\omega) \cup \text{poles}(\omega)$ any fixed point. The idea is to cut out a rectangle around each pole $\infty_j \in \text{poles}(\omega)$ and replace it by a one-handle. Indeed, choose a small

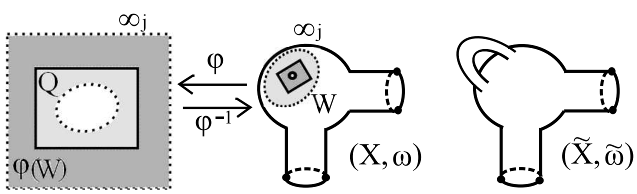

Figure 6:

topological disc W around ∞_j and map it to $\mathbb{C}$ by $\varphi(p) = \int_{q_0}^{p} \omega$ where p varies in W and $q_0 \in W$ is fixed. Notice, φ is well defined as the residue at ∞_j is 0, so the path of integration is irrelevant. The image $\varphi(W) \subset \mathbb{C}$ is the complement of a compact set (the total shaded region on figure 6 stretching to infinity). Draw a rectangle $Q \subset \varphi(W)$ as shown on figure 6 and remove its exterior (the darker region). On the surface, we remove the darker rectangular domain containing ∞_j. Then glue together the lower horizontal edge of Q to the upper and the left to the right, like gluing a torus. The gluing maps are clearly a vertical and a horizontal translation respectively. Therefore we obtain a handle with a translation structure compatible with the structure on the rest of the surface (see figure 6). By doing this for each ∞_j, we obtain a compact translation surface $(\tilde{X}, \tilde{\omega})$ of $\text{genus}(\tilde{X}) = \text{genus}(X) + \sharp(\text{poles}(\omega))$, where $\tilde{\omega}$ is now holomorphic (has no poles). A lot is known about the behavior of the geodesics on such surfaces [2], [4], [7], so we use this knowledge in our advantage. Let $\tilde{\Lambda}_{p_0}$ be the set of all directions $\theta \in \mathrm{S}^1$ for which the geodesic $\tilde{l}_{p_0}(\theta)$ on $\tilde{X}$ is closed or hits a zero of $\tilde{\omega}$. Also, let $\tilde{\Xi}$ be the set of all directions $\theta \in \mathrm{S}^1$ for which the geodesic flow of $(\tilde{X}, \tilde{\omega})$ in direction of θ is minimal [4] (e.g. an ergodic flow is minimal [2],[4]). Then $\tilde{\Lambda}_{p_0}$ is countable but dense in S^1 (see [7]) and $\tilde{\Xi}$ is dense and of full measure in S^1 (see [4], [2]). As a result, the set $\tilde{\Theta}_{p_0} = \tilde{\Xi} \setminus \tilde{\Lambda}_{p_0}$ consists of all $\theta \in \mathrm{S}^1$ for which the geodesic ray $\tilde{l}_{p_0}(\theta)$ is dense in $\tilde{X}$. Moreover, $\tilde{\Theta}_{p_0}$ is dense and of full measure in S^1. Therefore, for any $\theta \in \tilde{\Theta}_{p_0}$ the corresponding geodesic ray $l_{p_0}(\theta)$ on the original surface (X, ω) hits a pole of ω.

Let $U_{p_0} \subset \mathrm{S}^1$ be the set of all directions $\theta \in \mathrm{S}^1$ with the property that the geodesic ray $l_{p_0}(\theta)$ on (X, ω) in the direction of θ reaches a pole of ω. Since the geodesic flow on (X, ω) depends continuously on the initial point and direction, the condition that a geodesic ray reaches a planar infinity is open. Therefore, for each $\theta \in U_{p_0}$ there exists an open circular interval $(\alpha, \beta) \subset U_{p_0}$ that contains θ and for any $\theta' \in (\alpha, \beta)$ the ray $l_{p_0}(\theta')$ also reaches the same infinity. Hence, U_{p_0} is open in S^1. Moreover, the dense set of full measure $\tilde{\Theta}_{p_0}$ is contained in U_{p_0}. Therefore, U_{p_0} is open and dense set of full measure in S^1.

The second part of theorem 1 follows from the first one. If we are given a rational mirror configuration, unfold it into a translation surface with planar infinities (X, ω) as described earlier. Then, the infinity of the mirror domain lifts to the set of poles of ω on X and we apply the first part of the theorem.

Proof of theorem 2. As an open dense subset of S^1, the constructed U_{p_0} is a countable disjoint union of open circular intervals $(\alpha_j, \beta_j) \subset \mathrm{S}^1$, i.e. $U_{p_0} = \sqcup_{j=1}^{\infty} (\alpha_j, \beta_j)$. By construction, the geodesic rays $l_{p_0}(\theta)$ emitted from p_0 in all directions $\theta \in (\alpha_j, \beta_j)$ go to the same pole of ω. Fix some j and take a subinterval $(\alpha^*, \beta^*) \subseteq (\alpha_j, \beta_j)$ (it may even be convenient to choose $(\alpha^*, \beta^*) = (\alpha_j, \beta_j)$). Choose (α^*, β^*) so that its measure is less than π. Notice, that for every $\theta \in (\alpha^*, \beta^*)$, each ray $l_{p_0}(\theta)$ on X goes to the same $\infty^* \in \text{poles}(\omega)$. In particular, $\infty^* = \infty_3$ on figure 7. As $\sharp(\text{poles}(\omega)) \geq 2$, take another $\infty \in \text{poles}(\omega) \setminus \{\infty^*\}$ and call it ∞_1 just like on our picture below. Choose a "small" topological disc W around ∞_1 with the property $W \cap (\text{zeroes}(\omega) \cup \text{poles}(\omega)) = \{\infty_1\}$. Define the translation chart $\varphi(p) = \int_{q_0}^{p} \omega$, where p varies in W and $q_0 \in W$ is fixed. The zero residue at ∞_1 guaranties independence of the integral on the path between q_0 and p in W. On figure 7 we have also provided an analogous chart ψ around p_0. From now on, we use the same notations in W as the ones in $\varphi(W)$. Thus, we identify W with $\varphi(W)$. In $\mathbb{C}$ the domain W looks like the complement of a compact set (the shaded region on figure 7). Let $K \subset W$ be a Euclidean circle in $\mathbb{C}$ centered at O and containing $\mathbb{C} \setminus W$ in its interior.

Abusing notation, let α^* and β^* be the two points on the circle K such that the counter-clockwise angles between the positive horizontal line through O in $\mathbb{C}$ and the radii $O\alpha^*$ and $O\beta^*$ are respectively α^* and β^*. Let points T_1 and T_2 on K be such that counter-clockwise $\measuredangle\alpha^* OT_1 = \measuredangle T_2 O\beta^* = \frac{\pi}{2}$. Draw the lines t_1 and t_2 tangent to circle K at T_1 and T_2 respectively. Then they bound an infinite sector C, depicted on figure 7 as a darker shaded region.

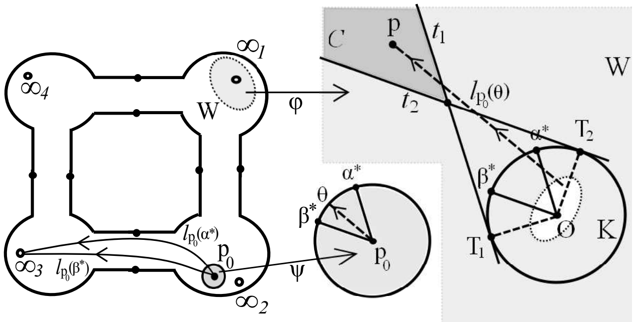

Figure 7:

We claim that that $C \subset X$ is not illuminated by p_0. Assume that for some point $p \in C$ there exists $\theta \in \mathrm{S}^1$ such that the geodesic $l_{p_0}(\theta) \subset X$ staring from p_0 in the direction of θ passes through p. Then, clearly $l_{p_0}(\theta)$ goes to ∞_1. As already commented in the introduction, any smooth geodesic on a translation surface forms the same angle with the horizontal direction at every point it passes through. In particular, the angle between $l_{p_0}(\theta)$ and the horizontal direction in the chart W as well as near the point p_0 is always θ. By looking at the picture of the chart W on figure 7, we see that $\theta \in (\alpha^*, \beta^*)$ in W. Hence $\theta \in (\alpha^*, \beta^*) \subset \mathrm{S}^1$ at the point p_0 as well. By the choice of the circular interval (α^*, β^*), the geodesic ray $l_{p_0}(\theta)$ should go to $\infty^* \neq \infty_1$. But a geodesic ray can only reach one pole of ω, so we get to a contradiction. Therefore, the infinite sector C on (X, ω) is not illuminated by $p_0 \in X$.

To conclude the proof, notice that for each circular interval $(\alpha^*, \beta^*) \subset U_{p_0}$ the unilluminated sector C near ∞_1 can be also constructed around any other pole $\infty \neq \infty^*$ of ω, i.e. there are $k-1$ unilluminated copies of C. Partition U_{p_0} into disjoint subintervals for which we can apply the construction of unilluminated infinite sectors from the preceding two paragraphs. Thus, the the total sum of the angles of all unilluminated sectors constructed on (X, ω) is $k-1$ times the total measure of $U_{p_0} \subset \mathrm{S}^1$ which is 2π. Hence, the total angle is $2\pi(k-1)$.

Proof of theorem 3. Let $D \subset \mathbb{C}$ be a rational mirror domain and $p_0 \in D$ (see figure 1 or 3). Recall the finite group G generated by all reflections in the lines through $0 \in \mathbb{C}$ parallel to the mirrors. It acts on S^1 by rotations. Let $f_{p_0} : U_{p_0} \to \mathrm{S}^1$ be the map described at the end of subsection "Main results" (see also figure 3) and assume it is not injective. Then, there are $\theta_1 \neq \theta_2$ from U_{p_0} such that $f_{p_0}(\theta_1) = f_{p_0}(\theta_2)$. Take the finite orbit $G(\theta_1) = \{g(\theta) \in \mathrm{S}^1 \mid g \in G\}$. Then $\theta \in G(\theta_1)$ if and only if $f_{p_0}(\theta) \in G(\theta_1)$ so $\theta_2 \in G(\theta_1)$. Hence, the restriction $f_{|_{G(\theta_1)}} : G(\theta_1) \to G(\theta_1)$ is not bijective and there is $\theta^* \in G(\theta_1)$ such that $\theta^* \in U_{p_0} \setminus f_{p_0}(U_{p_0})$. Since f_{p_0} is a restriction of a rotation on each connected component of U_{p_0}, there is $(\alpha^*, \beta^*) \ni \theta^*$ such that $(\alpha^*, \beta^*) \subset U_{p_0} \setminus f_{p_0}(U_{p_0})$. Remember the circle K from figure 3 that encompasses the mirrors and p_0. Using the circular interval (α^*, β^*), we can carry out absolutely the same construction as the one in the chart W described in the proof of theorem 2. For a picture of this construction look at the rightmost large shaded area W on figure 7. Observe that the notations of the current proof match the picture's notations so that we can use it directly, thinking that the set of mirrors is in the little white elliptic region containing the center O. We claim that the infinite sector C (the darker shaded area) is not illuminated by the source $p_0 \in D$. Indeed, assume there is a light ray emitted by p_0 that reaches some $p \in C$. Then, from the picture, the direction of this ray is $\theta \in (\alpha^*, \beta^*)$. But the light ray started from p_0 in some direction $\theta_0 \in \mathrm{S}^1$, so $\theta = f_{p_0}(\theta_0)$ which is a contradiction.

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