Approximating a Motorcycle Graph by a Straight Skeleton

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Abstract

We investigate how a straight skeleton can be used to approximate a motorcycle graph. We explain how to construct a planar straight-line graph G such that the straight skeleton of G reveals the motorcycle graph of M, for every given finite set M of motorcycles. An application of our construction is a proof of the P-completeness of the construction problem of straight skeletons of planar straight-line graphs and simple polygons with holes.

1 Introduction

1.1 Motivation

The straight skeleton $\mathcal{S}(G)$ of an n-vertex planar straight-line graph G is a skeleton structure similar to Voronoi diagrams, but consists of straight-line segments only. It was introduced by Aichholzer et al. [1] for polygons, and was generalized to planar straight-line graphs by Aichholzer and Aurenhammer [2]. The motorcycle graph problem was introduced by Eppstein and Erickson [5] in an attempt to extract the essential subproblem of constructing straight skeletons. Indeed, the algorithms by Cheng and Vigneron [4] and Huber and Held [6] use motorcycle graphs as a tool for the construction of straight skeletons.

In this work we reverse the question and ask how the motorcycle graph can be computed by employing straight skeletons: We show that for a given set of motorcycles we can construct a planar straight-line graph G such that certain parts of $\mathcal{S}(G)$ approximate the motorcycle graph up to a given tolerance. Since Eppstein and Erickson [5] proved that the motorcycle graph construction problem is P-complete, we can provide a proof that constructing the straight skeleton for planar straight-line graphs and polygons with holes is P-complete as well. As a consequence, there is no hope to find an efficient parallel algorithm for straight skeletons, unless $\mathsf{P} = \mathsf{NC}$.

We note that Eppstein and Erickson were the first to mention the P-completeness of straight skeletons: In [5] they claim that arguments similar to those used for proving the P-completeness of motorcycle graphs would apply to straight skeletons as well. Further, they suggest that a motorcycle graph can be approximated by the straight skeleton of a set of sharp isosceles triangles, with one triangle per motorcycle. No details and no proof are given, though. In this paper we pick up their suggestion and prove that the straight skeleton of a set of meticulously chosen triangles does indeed approximate the motorcycle graph of the motorcycles given.

1.2 Preliminaries and Definitions

Consider a planar straight-line graph G with n vertices, none of them being isolated. Vertices of degree one are called terminals. According to [2], the definition of the straight skeleton $\mathcal{S}(G)$ of G is based on a wavefront-propagation process: Each edge e of G sends out two wavefronts, which are parallel to e and have unit speed. At terminals of G an additional wavefront orthogonal to the single incident edge is emitted. The wavefront $\mathcal{W}(G,t)$ of G at some time t can be interpreted as a 2-regular kinetic straight-line graph. Except for the vertices originating from the terminals of G, all vertices of $\mathcal{W}(G,t)$ move along bisectors of straight-line edges of G, see Fig. 1. During the propagation of $\mathcal{W}(G,t)$ topological changes occur: a wavefront edge may collapse ("edge event") or a wavefront edge may be split by a wavefront vertex ("split event"). The straight-line segments traced out by the vertices of $\mathcal{W}(G,t)$ form $\mathcal{S}(G)$. The edges of $\mathcal{S}(G)$ are called "arcs" and bound the "faces" of $\mathcal{S}(G)$. Wavefront vertices are called reflex (resp. convex) if they have a reflex (resp. convex) angle at the side where the propagate to. The arcs of $\mathcal{S}(G)$ which are traced out by reflex (resp. convex) vertices of $\mathcal{W}(G,t)$ are called reflex (resp. convex) arcs.

Consider a set of moving points in the plane, called

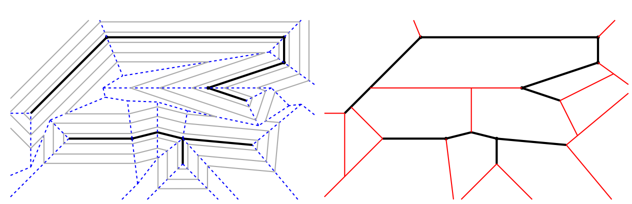

Figure 1: Left: The straight skeleton (dotted) is defined by propagating wavefronts (gray) of the input G (bold). Right: The motorcycle graph (red) induced by G.

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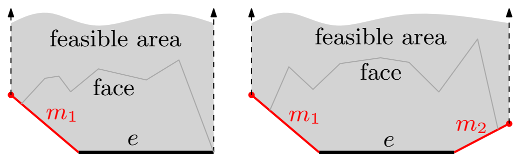

Figure 2: The feasible area of a face. Left: only m_1 has e as an arm. Right: m_1 and m_2 have e as an arm.

"motorcycles", that drive along straight-line rays according to given speed vectors. Each motorcycle leaves a trace behind it and stops driving — it "crashes" — when reaching the trace of another motorcycle. The arrangement of these traces is called motorcycle graph, cf. [5]. We denote by $\mathcal{M}(m_1, \ldots, m_n)$ the motorcycle graph of the motorcycles $m_1, \ldots, m_n$ where each motorcycle m_k is given by a start point p_k and speed vector v_k. We adopt the assumption by Cheng and Vigneron [4] that no two motorcycles may crash simultaneously.

In [6] we defined a motorcycle graph on a planar straight-line graph G, by generalizing Cheng and Vigneron's concept [4]. The idea is that a motorcycle m is defined for each reflex wavefront vertex v in $\mathcal{W}(G, 0)$ that starts from v and has the same speed vector as v. We call the wavefront edges incident to v the arms of m; the arm left (resp. right) of the speed ray of m is called left (resp. right) arm. Additionally, we assume that motorcycles crash if they hit an edge of G. We denote the resulting motorcycle graph by $\mathcal{M}(G)$, see Fig. 1. The following two theorems were proved in [4] for non-degenerate[1] polygons with holes and in [6] for general planar straight-line graphs G. (The concept of the feasible area of an edge e of G is illustrated in Fig. 2.)

Theorem 1 *$\mathcal{M}(G)$ covers the reflex arcs of $\mathcal{S}(G)$.*

Theorem 2 *Consider a face $f(e)$ of $\mathcal{S}(G)$ corresponding to the wavefront edge e. Then $f(e)$ is contained in a "feasible area" which is bounded by (i) e at time zero, (ii) by traces of motorcycles m that have e as an arm and (iii) by rays perpendicular to e starting the end of those motorcycle traces, if existing, and at the end of e otherwise.*

2 Approximating a motorcycle graph by a straight skeleton

Let us consider n motorcycles $m_1, \ldots, m_n$, where each motorcycle m_i has a start point p_i and a speed vector v_i. We assume that no two motorcycles crash simultaneously into each other. Can we find an appropriate planar straight-line graph G such that (a subset of the reflex arcs of) $\mathcal{S}(G)$ and $\mathcal{M}(m_1, \ldots, m_n)$ cover each other up to some given tolerance?

[1]A polygon is called non-degenerate if no two of the resulting motorcycles crash simultaneously into each other.

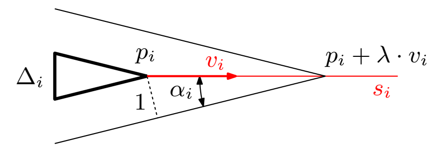

Figure 3: At each point p_i we set an isosceles triangle Δ_i with an angle of $2\alpha_i$, such that $\lambda|v_i| = 1/\sin\alpha_i$ and the motorcycle trace s_i bisects the angle of Δ_i at p_i.

Theorem 1 tells us that the reflex arc of $\mathcal{S}(G)$ that corresponds to a motorcycle m_i approximates the trace of m_i up to some gap. One observes that the faster m moves the better its trace tends to be approximated by the corresponding reflex arc in $\mathcal{S}(G)$. It is easy to see that $\mathcal{M}(m_1, \ldots, m_n)$ remains unchanged if we multiply each speed vector v_i by a positive constant λ. In order to obtain reflex arcs of $\mathcal{S}(G)$ that overlap with the traces in $\mathcal{M}(m_1, \ldots, m_n)$ we put at each start point p_i an isosceles triangle Δ_i with an angle of $2\alpha_i$ at p_i, where

$$\alpha_i := \arcsin \frac{1}{\lambda|v_i|}, \tag{1}$$

and λ large enough such that $\lambda|v_i| \geq 1$ for all $1 \leq i \leq n$, see Fig. 3. The lengths of the arms of Δ_i will be specified later.

By definition of $\mathcal{S}(G)$, a reflex wavefront vertex u_i is emanated from each p_i and its speed vector equals λ-times the speed vector of the motorcycle m_i. However, note that further motorcycles are introduced at the additional corners of each triangle. We can now rephrase our initial question: can we always find λ large enough such that those reflex arcs that are traced out by u_i approximate $\mathcal{M}(m_1, \ldots, m_n)$ up to some given tolerance?

Let us denote by s_i the trace of m_i and by $s_i + D_r$ the Minkowski sum of s_i and the disk with radius r centered at the origin. The motorcycle traces of $\mathcal{M}(m_1, \ldots, m_n)$ are closed sets and two traces s_i, s_j intersect if and only if m_i crashed into m_j or vice versa. Hence there is an $\mu > 0$ such that for all $1 \leq i < j \leq n$ it holds that m_i crashes into m_j or vice versa if and only if $s_i + D_\mu$ intersects $s_j + D_\mu$. For example, let μ be a third of the minimum of the pairwise infimum distances of disjoint traces s_i, s_j. We further assume that μ is small enough such that $p_i + D_\mu$ does not intersect any $s_j + D_\mu$ for all $1 \leq i, j \leq n$, except if $p_i \in s_j$. (This will be needed for Lemma 4.)

We define the planar straight-line graph G by the set of triangles Δ_i at each start point p_i, where the length of the arms incident to p_i are set to $\mu/2$ and the angle of Δ_i at p_i is given by Eqn. (1).

Lemma 3 *For any $1 \leq i \leq n$ the wavefronts of Δ_i stay within $s_i + D_\mu$ until time $\mu/4$ for $\lambda \geq \frac{2}{|v_i|}$.*

Proof. Consider a triangle Δ_i at some start point p_i. Note that λ is large enough such that $2\alpha_i$ is at most

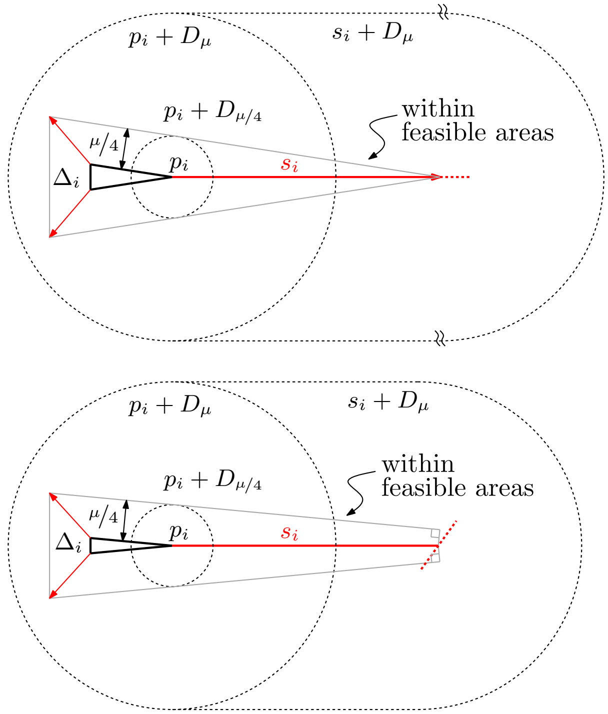

Figure 4: The wavefronts of Δ_i are bounded to $s_i + D_\mu$ until time $\mu/4$. Top: the motorcycle m_i did not crash until that time. Bottom: the motorcycle m_i did crash into another trace (dotted line segment) until that time.

60^o. Hence, the other two angles of Δ_i are at least 60^o and, therefore, the two additional motorcycles at Δ_i have a speed of at most 2. Since the start points of those motorcycles are $\mu/2$ away from p_i and they drive at most a distance of $\mu/2$ in time $\mu/4$, they stay within $p_i + D_\mu$.

According to Thm. 2, we consider the feasible areas of the edges of Δ_i restricted to an orthogonal distance of at most $\mu/4$ to the edges of Δ_i, see Fig. 4. We distinguish two cases: the motorcycle m_i (i) did crash or (ii) did not crash until time $\mu/4$. However, in both cases the corner points of these restricted feasible areas are contained within $s_i + D_\mu$, the restricted feasible areas are convex and $s_i + D_\mu$ is convex. Hence, the wavefronts of Δ_i until time $\mu/4$ are contained within $s_i + D_\mu$. □

We denote by $\overrightarrow{s_i}$ the ray starting at p_i in direction v_i and define

$$L := \max_{1 \le i,j \le n} d(p_i, \overrightarrow{s_i} \cap \overrightarrow{s_j}). \tag{2}$$

Note that we may only consider indices i, j for which $\overrightarrow{s_i} \cap \overrightarrow{s_j}$ is not empty. If no such indices i, j exist then we set L to zero. Further, let us denote by $\varphi_{i,j} \in [0, \pi]$ the non-oriented angle spanned by v_i and v_j, with $\varphi_{i,j} = \varphi_{j,i}$. Next we define

$$\Phi := \min_{1 \le i < j \le n} \mathbb{R}^+ \cap \{\varphi_{i,j}, \pi - \varphi_{i,j}\}. \tag{3}$$

If this set it is empty, i.e. if all motorcycles are driving on parallel tracks, then we set $\Phi := \pi/2$.

Lemma 4 *Let m_i denote a motorcycle crashing into the motorcycle m_j. The wavefronts of Δ_i do not cause a split event for u_j until time $\mu/4$ for $\lambda \ge \frac{2}{\min_k |v_k| \sin \Phi}$.*

Proof. We note that $\lambda \ge \frac{2}{|v_k| \sin \Phi}$ holds for any $1 \le k \le n$ which means that

$$\sin \alpha_k \le \frac{1}{|v_k|\lambda} \le \frac{1}{2} \sin \Phi \le \sin \frac{\Phi}{2},$$

since sin is concave on $[0, \pi]$. By further noting that sin is monotone on $[0, \pi/2]$ we see that

$$\alpha_k \le \frac{\Phi}{2} \qquad \forall\, 1 \le k \le n.$$

The case where m_j also crashes into m_i is excluded since two motorcycles do not crash simultaneously by assumption. The cases where s_i and s_j are collinear are either trivial or excluded by assumption. Without loss of generality, we may assume that s_i is right of $\overrightarrow{s_j}$, see Fig. 5. We denote by q the endpoint of the reflex straight-skeleton arc incident to p_i. Let us consider the left (resp. right) bisector between the left (resp. right) arm of m_i and the right arm of m_j, starting from q.

From the proof of Lem. 3 it follows that in order that u_j is involved in a split event with a wavefront from Δ_i until time $\mu/4$ it is necessary that one of both bisectors intersects $\overrightarrow{s_j}$. (Check Figure 4: The two additional motorcycles from Δ_i stay within $p_i + D_\mu$. Hence, we only have to consider the arms of m_i.) Let us consider the right bisector. Recall that $\alpha_i, \alpha_j \le \Phi/2$ and that $\pi - \varphi_{i,j} \ge \Phi$. In the extremal case, where equality is attained for all three inequalities, the right bisector is just parallel to s_j, but strictly right of $\overrightarrow{s_i}$. In all other cases the bisector rotates clockwise at q such that our assertion is true in general. Analogous arguments hold for the left bisector. Summarizing, the vertex u_j does not lead to a split event with the wavefronts of Δ_i until time $\mu/4$. □

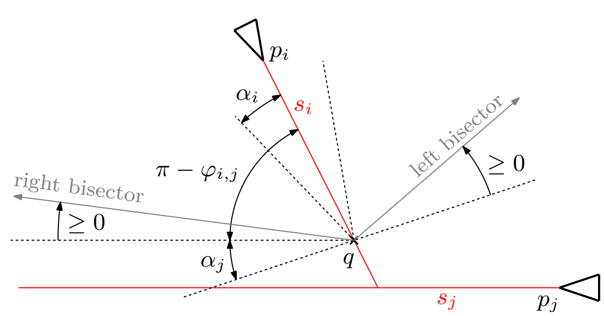

Figure 5: The wavefronts of Δ_i do not cause a crash with u_j.

Lemma 5 *Let m_i denote a motorcycle crashing into the motorcycle m_j. For any $\epsilon > 0$ and*

$$\lambda \geq \frac{1}{\min_k |v_k| \cdot \sin \Phi} \cdot \max\left\{2, \frac{L}{\min\{\mu/4, \epsilon\}}\right\},$$

the trace s_i is covered up to a gap size ϵ by the reflex arc traced out by u_i.

Proof. We will prove the following: consider an arbitrary point q on s_i whose distance to the endpoint of s_i is at least ϵ. Then we show that q is reached by u_i until time $\mu/4$.

Consider Fig. 6. We first show that until time $\mu/4$ the vertex u_i may only cause a split event with the wavefronts of Δ_j. By Lem. 3, we know that until time $\mu/4$ only the wavefronts of a triangle Δ_k could cause a split event with u_i if s_k and s_i intersect. Hence, m_k crashed against s_i. However, by Lem. 4 it follows that u_i does not lead to a split event with the wavefronts from Δ_k.

W.l.o.g., we may assume that s_i lies right to $\overrightarrow{s_j}$. To show that u_i reaches q until time $\mu/4$ it suffices to prove that q has a smaller orthogonal distance to the left arm of m_i than to the right arm of m_j and that the orthogonal distance of q to the left arm of m_i is at most $\mu/4$.

The orthogonal distance of q to the left arm of m_i is at most $L \cdot \sin \alpha_i$. The orthogonal distance of q to the right arm of m_j is at least the orthogonal distance of q to s_j. However, this distance is at least $\epsilon \cdot \sin \varphi_{i,j}$. Summarizing, our assertion holds if

$$L \cdot \sin \alpha_i \leq \min\{\mu/4, \epsilon \sin \varphi_{i,j}\},$$

which is

$$\lambda \geq \frac{L}{|v_i| \cdot \min\{\mu/4, \epsilon \sin \varphi_{i,j}\}}.$$

However, our choice for λ fulfills this condition. The case where s_j and s_i are collinear such that m_i crashes at p_j is similar. The wavefront of Δ_j reaches q at least in time $\epsilon/2$ and v_i reaches q in at most $\frac{L}{\lambda|v_i|}$ time. □

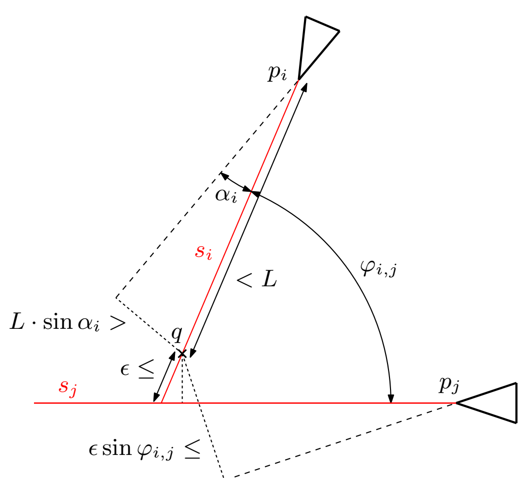

Figure 6: The point q has a smaller orthogonal distance to the left arm of m_i than to the right arm of m_j.

Let us define by $\mathcal{S}^*_\lambda(m_1, \ldots, m_n) \subset \mathcal{S}(G)$ the union of the reflex straight-skeleton arcs which emanate from $p_1, \ldots, p_n$, where G is given as described above. Then we get the following corollary of Lem. 5.

Corollary 6

$$\lim_{\lambda \to \infty} \mathcal{S}^*_\lambda(m_1, \ldots, m_n) = \mathcal{M}(m_1, \ldots, m_n)$$

This corollary also asserts that a point q on a motorcycle trace s_i of a motorcycle m_i that never crashed is covered by an arc of $\mathcal{S}^*_\lambda(m_1, \ldots, m_n)$ for large enough λ. However, this is easily proved by applying Lem. 3 and Lem. 4, and by finally finding λ large enough such that the point q is reached by u_i until time $\mu/4$.

3 Computing the motorcycle graph

In order to actually compute the motorcycle graph $\mathcal{M}(m_1, \ldots, m_n)$ from $\mathcal{S}^*_\lambda(m_1, \ldots, m_n)$ for some big λ, we still have to cope with the remaining gaps within $\mathcal{S}^*_\lambda(m_1, \ldots, m_n)$. For deciding whether a motorcycle m_i actually escapes or crashes into some trace s_j we want to determine λ large enough such that the following two conditions hold:

- If m_i crashes into a trace s_j then u_i leads to a split event until the time $\mu/4$ and the reflex arc traced out by u_i has an endpoint in a straight-skeleton face of an edge of Δ_j. (The right arm of m_j if s_i is right of $\overrightarrow{s_j}$ and the left arm if s_i is left of $\overrightarrow{s_j}$.)
- If m_i escapes then u_i did not lead to a split event until the time $\mu/4$.

Lemma 7 *Consider $\mathcal{S}(G)$ with*

$$\lambda \geq \frac{\max\left\{2, \frac{8L}{\mu}\right\}}{\min_k |v_k| \cdot \sin \Phi}.$$

Then m_i crashes into s_j if and only if u_i leads to a split event with a wavefront emanated by Δ_j until time $\mu/4$. In particular, m_i escapes if and only if u_i does not lead to a split event until time $\mu/4$.

Proof. We distinguish two cases. First, suppose that the motorcycle m_i crashed into the trace s_j, see Fig. 7. We may assume without loss of generality that s_i is right of $\overrightarrow{s_j}$. First we note that by our choice of λ we may apply Lem. 3. We denote by p the intersection $s_i \cap s_j$. Further, we set $\epsilon := \mu/8$ which allows us to apply Lem. 5 since

$$\max\left\{2, \frac{8L}{\mu}\right\} \geq \max\left\{2, \frac{L}{\min\{\mu/4, \epsilon\}}\right\}.$$

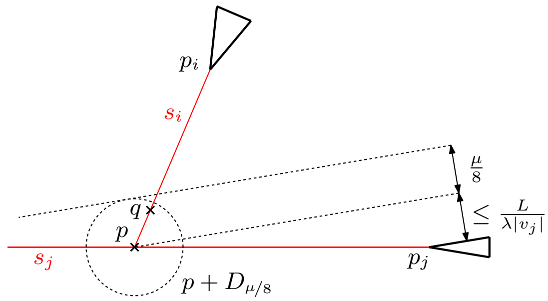

Figure 7: The reflex wavefront vertex at p_i is forced to cause a split event until time $\mu/4$.

Thus, the endpoint q of the reflex arc traced out by u_i has a distance of at most $\mu/8$ to p.

On the other hand, u_j reaches p in at most $\frac{L}{\lambda|v_j|}$ time. We conclude that the wavefront edge from the right arm of m_j reaches q in at most $\frac{L}{\lambda|v_j|} + \frac{\mu}{8}$ time which is bounded from above by

$$\frac{L \cdot \mu \cdot \min_k |v_k| \cdot \sin \Phi}{8 \cdot L \cdot |v_j|} + \frac{\mu}{8} \leq \frac{\mu}{4}$$

by our choice of λ. Summarizing, the point q is swept by the wavefront of the right arm of m_j and is reached by u_i until $\mu/4$ time. Hence, the vertex u_i must have caused a split event until the requested time by crashing into the wavefront of the right arm of m_j.

For the second case assume that m_i escapes. Lemmas 3 + 4 imply that u_i does not lead to a split event until time $\mu/4$. □

In order to compute the motorcycle graph by employing a straight skeleton algorithm, we would have to compute the appropriate values for L, Φ, μ in order to determine a sufficiently large λ. While L and Φ are already given independent of $\mathcal{M}(m_1, \ldots, m_n)$, the following lemma gives a formula for μ for which the actual motorcycle graph is not needed to be known. (In the following lemma we take $d(\overrightarrow{s_i}, \emptyset)$ to be infinity.)

Lemma 8 *For any two disjoint motorcycle traces s_i and s_j the Minkowski sums $s_i + D_\mu$ and $s_j + D_\mu$ are disjoint if*

$$\mu := \frac{1}{3} \min_{1 \leq i,j,k \leq n} \mathbb{R}^+ \cap \{d(\overrightarrow{s_i}, p_j), d(\overrightarrow{s_i}, \overrightarrow{s_j} \cap \overrightarrow{s_k})\}.$$

Proof. In order to guarantee that the Minkowski sums are disjoint, it suffices to set μ to a lower bound of a third of the minimum of all pairwise infimum distances of disjoint traces s_i and s_j.

Let us consider two disjoint traces s_i and s_j. We choose two points $q_i \in s_i, q_j \in s_j$ for which $d(s_i, s_j) = d(q_i, q_j)$ holds. We may assume that either q_i is an endpoint of s_i or q_j is an endpoint of s_j. (If s_k is a ray, the only endpoint is p_k.) If q_j is the start point of s_j then we have $d(s_i, s_j) = d(s_i, p_j) \geq 3\mu$. If q_j is the opposite endpoint of s_j — and hence s_j is a segment — then s_j crashed into some other motorcycle trace. So there is a trace s_k such that $q_j = s_j \cap s_k$. Again we get $d(s_i, s_j) = d(s_i, q_j) \geq 3\mu$. Analogous arguments hold if q_i is an endpoint of s_i. □

After computing appropriate values for L, Φ and μ for a set of motorcycles $m_1, \ldots, m_n$, we can determine a sufficiently large λ and build the input graph G by constructing the triangles $\Delta_1, \ldots, \Delta_n$ as described. After computing the straight skeleton $\mathcal{S}(G)$ we determine the length of each motorcycle's trace by applying the conditions listed in Lemma 7.

4 Constructing the straight skeleton is P-complete

Atallah et al. [3] described a framework for reductions of the P-complete PLANAR CIRCUIT VALUE problem and used it to prove the P-completeness of several geometric problems. However, investigating the P-completeness of geometric problems often requires the availability of exact geometric computations which are not in NC. In order to investigate the P-completeness of geometric problems Attalah et al. [3] propose that the answers to basic geometric queries are provided by an oracle.

A basic building block for showing that the straight skeleton is P-complete is the construction of the triangles Δ_i. Assume p_i, α_i, v_i, μ, and λ are given. We further assume that an oracle determines the intersection points of two circles with given centers and radii. Then we can construct Δ_i as follows. We first compute the point $q_i = p_i + \lambda v_i$, which is the position of m_i at time one, see Figure 3. Then we construct the circle C_1 with $[p_i, q_i]$ as diameter and the circle C_2 centered at p_i with radius 1. The two circles C_1, C_2 intersect at two points, say a_i, b_i. The triangle Δ_i^* with vertices a_i, b_i, q_i is an isosceles triangle with angle $2\alpha_i$ at q_i and therefore similar to Δ_i. The length of the arms of Δ_i^* at q_i are at most $\lambda|v_i|$. By scaling the triangle by the factor $\mu/2\lambda|v_i|$ and by translating it accordingly, we get a triangle with the desired geometry. (Actually, the arms of the constructed triangle are a bit shorter than $\mu/2$, but this is only to our advantage.)

Eppstein and Erickson [5] proved that the computation of the motorcycle graph is P-complete by presenting a LOGSPACE reduction of the CIRCUIT VALUE problem to the computation of the motorcycle graph. The CIRCUIT VALUE problem asks for the output value of a gate in a binary circuit with n input gates, which is presented an input vector of binary values. Eppstein and Erickson demonstrated how to translate the CIRCUIT VALUE problem to the motorcycle graph construction problem by simulating each gadget using motorcycles. The values 1 and 0 on a wire are represented by the presence or absence of a motorcycle on a track. The original question for the output value of a particular gate of the circuit can be translated to the question whether a specific motorcycle crashes until some distance from its start point. In other words, Eppstein and Erickson proved that the decision problem whether a specific motorcycle crashes until some distance from its start point is P-complete.

Lemma 9 *The construction of the straight skeleton of a planar straight-line graph is P-complete under LOGSPACE reductions.*

Proof. Eppstein and Erickson reduced the CIRCUIT VALUE problem to a specific motorcycle graph problem. The next step is to reduce the motorcycle graph problem to the straight-skeleton problem: we construct a suitable input graph G which allows us to apply Lem. 7 for deciding whether a specific motorcycle crashes until some distance from its start point.

According to [5] all $O(1)$ different types of motorcycle gadgets are arranged in an $n \times n$ grid, and each gadget takes constant space and consists of $O(1)$ motorcycles. For determining a sufficiently large λ we need bounds on L, Φ and μ. An upper bound on L is the length of the diagonal of the $n \times n$ grid. Further, $\sin \Phi \geq 1/2$ since the direction angles of the motorcycles are all multiples of $\pi/4$. A lower bound on μ can be found by considering each gadget independently and taking the minimum among them. Finally, we build G by constructing for each motorcycle (independently from each other) an isosceles triangle, as described in Section 2. □

We can easily extend the construction of G to form a polygon with holes, by adding a sufficiently large bounding box to G. As remarked in [5], only one motorcycle m may leave the bounding box B of the $n \times n$ grid. The motorcycle m encodes the output of the binary circuit by leaving B if the circuit evaluates to 1 and by crashing within B if the circuit evaluates to 0.

By Lemma 7, the reflex wavefront vertex u, which corresponds to m, encodes the output of the binary circuit by leading to a split event until time $\mu/4$ if and only if the circuit evaluates to 0. Lemma 3 implies that the wavefront vertices stay within $B + D_\mu$, except possibly u. Hence, we could enlarge B by 2μ at each side and add it to G such that the wavefronts of B do not interfere with the wavefronts of the triangles until time $\mu/4$, except for u. Still, we can determine the output of the binary circuit by checking whether the reflex straight-skeleton arc that corresponds to u ends within $B + D_\mu$ until time $\mu/4$. (Recall that the end of a reflex straight-skeleton arc marks the place where the reflex wavefront vertex led to a split event.)

Corollary 10 *The construction of the straight skeleton of a polygon with holes is P-complete under LOGSPACE reductions.*

Unfortunately, our P-completeness proof cannot be applied easily to simple polygons. Consider the five motorcycles depicted in Fig. 8. A polygon, whose reflex straight-skeleton arcs would approximate the motorcycle traces, would need to connect the start points of m and $m_1, \ldots, m_4$. But in order to decide where the red square formed by the traces of $m_1, \ldots, m_4$ can be penetrated by the polygon, while avoiding to stop a motorcycle too early, it would be necessary to know a specific pair m_i, m_j of motorcycles such that m_i is guaranteed to crash into the trace of m_j. However, deciding whether a specific motorcycle crashes does not seem much easier than computing the whole motorcycle graph. Hence, it remains open whether the computation of straight skeletons of polygons is P-complete.

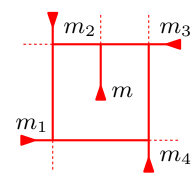

Figure 8: These motorcycle traces cannot be approximated easily by a straight skeleton of a simple polygon.

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